Network Synthesis and Realisability

12

CHAPTER OBJECTIVES

After carefully studying this chapter, you should be able to do the following:

- Explain the concept of network synthesis.
- Explain the Hurwitz conditions for stability.
- Test polynomials for stability applying Hurwitz stability criteria.
- State the properties of a positive real function.
- Test a network function to find whether it is a positive real function and hence realizable.
- Synthesize networks by Foster method.
- Synthesize networks by Cauer method.
- From the driving point impedance or admittance function realize the network in Foster and Cauer form I and II, respectively.

12.1 INTRODUCTION

Network analysis deals with finding out the output response, using various techniques, when the excitation signal (input signal) and the network are known.
Network synthesis deals with the realisation of the network from the given excitation and output response.

The network synthesis provides alternate solutions. The network synthesis technique is used in filter design where computations, innovations and judgement are required. In the network synthesis, we first determine the driving point function in impedance and admittance form. The driving point admittance/immittance is given by the following form:

$$F(s) = \frac{R(s)}{E(s)}$$

where $R(s)$ is the output response and $E(s)$ is the excitation response.

There are two basic considerations to be taken care of. These are: (i) realisability and (ii) stability.

It is determined first whether the function is realisable or not; that is, to find out whether it is possible to obtain a passive network. This is termed as realisability. For stability, there are certain conditions that must be satisfied by the network function.

A network function is tested first for its realisability. It is then tested for its stability. After that, we proceed to realize the network using Foster and Cauer methods.

In this chapter, we will examine the network function for their stability condition. We will also explain the process of realisation of the network, when the driving point network function is given.

12.2 HURWITZ CONDITIONS FOR STABILITY

The driving point impedance or admittance is expressed as the ratio of two polynomials, $P(s)$ and $Q(s)$. These polynomials are tested for Hurwitz stability conditions. If they satisfy the conditions, then we say that they are Hurwitz.

Consider a polynomial:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0$$

Then, for the polynomial to be Hurwitz, it has to satisfy the following Hurwitz's conditions for stability:

Condition 1: All the coefficients $a_0, a_1, a_2 \ldots a_n$ must be positive.

Condition 2: No powers of 's' should be absent between the highest and lowest degree term of 's' unless all the even or all the odd terms are missing. For example, $P(s) = s^5 + 3s^3 + 2s^2 + 3s + 1$ is not Hurwitz as the term s^4 is missing. The polynomial $P(s) = s^5 + 2s^3 + 3s$ is Hurwitz as all coefficients are positive and all the even terms are missing.

Condition 3: The continued fraction expansion of the ratio of the odd to even parts or even to odd parts of Hurwitz polynomial gives all positive quotient terms.

Now, consider a polynomial $P(s) = s^4 + s^3 + 2s^2 + 3s + 2.$

The even parts of the polynomial $P(s)$, that is, $e(s) = s^4 + 2s^2 + 2$,

while the odd parts of the polynomial $P(s)$, that is, $e(o) = s^3 + 3s$.

By continued fraction expansion, if it is seen that all the quotient terms are positive, then the given polynomial $P(s)$ is Hurwitz.

If a polynomial satisfies the conditions of Hurwitz for stability, we say that the polynomial must be Hurwitz. The testing of a polynomial for satisfying the conditions of Hurwitz is further explained through examples.

Example 12.1 Check whether the given polynomial is Hurwitz or not

$$P(s) = s^4 + s^3 + 2s^2 + 3s + 2$$

Solution: Given

$$P(s) = s^4 + s^3 + 2s^2 + 3s + 2.$$

All the coefficients (that is, s^4, s^3, s^2 and s are positive). There are no terms missing between the highest and lowest degree terms. Both the conditions, that is, condition 1 and condition 2 are satisfied.

For condition 3 we will perform the continued fraction expansion.

Now, $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$

Even part of $P(s) = s^4 + 2s^2 + 2 = e(s)$

Odd part of $P(s) = s^3 + 3s = o(s)$

The continued fraction expansion of $\frac{e(s)}{o(s)}$ is given by the following:

$$
\begin{array}{l}
s^3 + 3s \overline{)s^4 + 2s^2 + 2} (s \\
\quad \underline{s^4 + 3s^2} \\
\quad\quad -s^2 + 2 \\
-s^2 + 2 \;)\, s^3 + 3s \,(-s \\
\quad\quad \underline{s^3 - 2s} \\
\quad\quad\quad 5s \;)\, -s^2 + 2 \,(-\frac{s}{5} \\
\quad\quad\quad\quad \underline{-s^2} \\
\quad\quad\quad\quad\quad 2 \;)\, 5s \,(\frac{5s}{2} \\
\quad\quad\quad\quad\quad\quad \underline{5s} \\
\quad\quad\quad\quad\quad\quad\quad \text{x}
\end{array}
$$

Since all the quotient terms are not positive, the given polynomial is not Hurwitz.

Example 12.2 Test if the polynomial $s^3 + 6s^2 + 12s + 8$ is Hurwitz.

Solution: Given

$$P(s) = s^3 + 6s^2 + 12s + 8$$

We have to test for stability with respect to all the three conditions namely

1. All the coefficients must be positive.
2. No powers of s should be absent and
3. Continued fraction expansion should give all positive quotient terms

The given polynomial satisfies the first two conditions. We will test for the third condition. Now, even part of $P(s) = 6s^2 + 8 = e(s)$ and odd part of $P(s) = s^3 + 12s = o(s)$

We will use the continued fraction expansion as follows:

$$\frac{o(s)}{e(s)} = \frac{s^3 + 12s}{6s^2 + 8}$$

$$6s^2 + 8 \overline{)\, s^3 + 12s \,} \left(\frac{s}{6} \right.$$

$$\underline{s^3 + \frac{4}{3}s}$$

$$\frac{32}{3}s \,) 6s^2 + 8 \left(\frac{9}{16}s \right.$$

$$\underline{6s^2}$$

$$8 \,) \frac{32}{3}s \left(\frac{4s}{3} \right.$$

$$\underline{\frac{32}{3}s}$$

$$\text{x}$$

Since all the quotient terms of $\frac{o(s)}{e(s)}$ are positive, the given polynomial is Hurwitz.

Example 12.3 Test if the polynomial $s^4 + 8s^2 + 32$ is Hurwitz.

Solution: In this case, only even part is given in the following:

$$e(s) = s^4 + 8s^2 + 32$$

To get odd part, we will differentiate even part $e(s)$ as follows:

$o(s) = e'(s) = 4s^3 + 16s$. We find continued fraction expansion of even to odd parts as

$$4s^3 + 16s \overline{)\, s^4 + 8s^2 + 32 \,} \left(\frac{s}{4} \right.$$

$$\underline{s^4 + 4s^2}$$

$$4s^2 + 32 \,) 4s^3 + 16s \,(s$$

$$\underline{4s^3 + 32s}$$

$$-16s \,) 4s^2 + 32 \left(-\frac{s}{4} \right.$$

$$\underline{4s^2}$$

$$32 \,) -16s \left(-\frac{s}{2} \right.$$

$$\underline{-16s}$$

$$\text{x}$$

Since all powers of s are not present and all the quotient terms are not positive, and the polynomial is not Hurwitz.

Example 12.4 Test whether the polynomial $s^5 + 5s^3 + 4s$ is Hurwitz.

Solution: In this case, only odd part is given.

That is, $o(s) = s^5 + 5s^3 + 4s$

To get the even part, we will differentiate the odd part.

That is, $e(s) = o'(s) = 5s^4 + 15s^2 + 4$

The continued fraction expansion is given by the following:

$$
\begin{array}{l}
5s^4 + 15s^2 + 4 \overline{)s^5 + 5s^3 + 4s} \left(\frac{s}{5} \right. \\
\quad \underline{s^5 + 3s^3} \\
\quad\quad 2s^3 + 4s \;\big)5s^4 + 15s^2 + 4 \left(\frac{5}{2}s \right. \\
\quad\quad\quad \underline{5s^4 + 10s^2} \\
\quad\quad\quad\quad 5s^2 + 4 \;\big)2s^3 + 4s \left(\frac{2}{5}s \right. \\
\quad\quad\quad\quad\quad \underline{2s^3 + \frac{8}{5}s} \\
\quad\quad\quad\quad\quad\quad \frac{12}{5}s \;\big)5s^2 + 4 \left(\frac{25}{12}s \right. \\
\quad\quad\quad\quad\quad\quad\quad \underline{5s^2} \\
\quad\quad\quad\quad\quad\quad\quad\quad 4 \;\big)\frac{12}{5}s \left(\frac{12}{20}s \right. \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{\frac{12}{5}s} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{x}
\end{array}
$$

Since all the quotient terms are positive, the polynomial is Hurwitz.

Example 12.5 Find the limits of K so that the polynomial $s^3 + 14s^2 + 56s + K$ may be Hurwitz.

Solution: By inspection, we see that the first two conditions are satisfied.

Odd part of the given polynomial, that is, $o(s) = s^3 + 56s$
Even part of the given polynomial, that is, $e(s) = 14s^2 + \text{K}$

The continued fraction expansion is given by the following:

$$14s^2 + K\overline{\left|\begin{array}{l} s^3 + 56s \\ s^3 + \dfrac{Ks}{14} \end{array}\right.}\left(\begin{array}{l} \dfrac{s}{14} \\ \left(56 - \dfrac{K}{14}\right)s \end{array}\right.\overline{\left|\begin{array}{l} 14s^2 + K \\ 14s^2 \end{array}\right.}\left(\begin{array}{l} \dfrac{14s}{56 - \dfrac{K}{14}} \\ K \end{array}\right.\overline{\left|\begin{array}{l} \left(56 - \dfrac{K}{14}\right)s \\ \left(56 - \dfrac{K}{14}\right)s \end{array}\right.}\left(\begin{array}{l} \dfrac{\left(56 - \dfrac{K}{14}\right)s}{K} \\ \text{x} \end{array}\right.$$

Now, for the polynomial to be Hurwitz, quotient terms should be positive.

That is, $$\frac{14}{56 - \dfrac{K}{14}} > 0$$

and $$\frac{56 - \dfrac{K}{14}}{K} > 0$$

or $$\frac{56 - \dfrac{K}{14}}{14} > \infty$$

or $$56 - \frac{K}{14} > 0$$

This gives no value for K.

or $$56 > \frac{K}{14}$$

or $$K < 56 \times 14$$

or $$K < 56 \times 14$$

or $$K < 784$$

Therefore, the limit of K is $0 < K < 784$.

Example 12.6 Test whether the following polynomial is Hurwitz or not.

$$F(s) = \frac{2s^4 + 6s^3 + 11s^2 + 10s + 5}{s^4 + 5s^3 + 8s^2 + 9s + 6}$$

Solution: Given

$$F(s) = \frac{2s^4 + 6s^3 + 11s^2 + 10s + 5}{s^4 + 5s^3 + 8s^2 + 9s + 2} = \frac{P(s)}{Q(s)}$$

Let us first check for $P(s)$:

$$P(s) = 2s^4 + 6s^3 + 11s^2 + 10s + 5$$

Even part of $P(s) = 2s^4 + 11s^2 + 5$

Odd part of $P(s) = 6s^3 + 10s$

The continued fraction expansion is given by the following:

$$6s^3 + 10s \overline{)2s^4 + 11s^2 + 5} \left(\frac{s}{3} \right.$$

$$\underline{2s^4 + \frac{10}{3}s^2}$$

$$\frac{23}{3}s^2 + 5 \overline{)6s^3 + 10s} \left(\frac{18s}{23} \right.$$

$$\underline{6s^3 + \frac{90}{23}s}$$

$$\frac{140}{23}s \overline{)\frac{23}{3}s^2 + 5} \left(\frac{23}{140}s \times \frac{23}{3}s^2 = \frac{529}{420}s \right.$$

$$\underline{\frac{23}{3}s^2}$$

$$5 \overline{)\frac{140}{23}s} \left(\frac{1}{5} \times \frac{140s}{23} = \frac{28s}{23} \right.$$

$$\underline{\frac{140}{23}s}$$

x

Since all the quotient terms of $P(s)$ are positive, $P(s)$ is Hurwitz.

Now, let us check for $Q(s)$:

$$Q(s) = s^4 + 5s^3 + 8s^2 + 9s + 6$$

Even part of $Q(s) = s^4 + 8s^2 + 6$ and Odd part of $Q(s) = 5s^3 + 9s$

The continued fraction expansion is carried out as follows:

$$5s^3 + 9s \overline{)s^4 + 8s^2 + 6} \left(\frac{1}{5}s \right.$$

$$\underline{s^4 + \frac{9}{5}s^2}$$

$$\frac{31}{5}s^2 + 6 \overline{)5s^3 + 9s} \left(\frac{5}{31s^2} \times 5s^3 = \frac{25}{31}s \right.$$

$$\underline{5s^3 + \frac{150}{31}s}$$

$$\frac{129}{31}s\overline{)\frac{31}{5}s^2+6}\left(\frac{31}{129s}\times\frac{31}{5}s^2=\frac{961s}{645}\right.$$

$$\frac{\frac{31}{5}s^2}{6\overline{)\frac{129}{31}s}}\left(\frac{1}{6}\times\frac{129s}{31}=\frac{129s}{186}\right.$$

$$\frac{\frac{129}{31}s}{\text{x}}$$

All the quotient terms of $Q(s)$ are also positive, and therefore, $Q(s)$ is also Hurwitz.

Since $P(s)$ and $Q(s)$ both are Hurwitz, $F(s)=\dfrac{P(s)}{Q(s)}$ is also Hurwitz.

12.3 PROPERTIES OF POSITIVE REAL FUNCTIONS

Positive real functions (PRF) represent physically realisable networks.

The driving point impedance function $Z(s)$ and driving point admittance function $Y(s)$ of a network can be expressed as the ratio of two polynomials.

If $F(s)=\dfrac{P(s)}{Q(s)}=\dfrac{a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{\text{n}}}{b_0s^m+b_1s^{m-1}+b_2s^{m-2}+\cdots+b_{\text{m}}}$ is an immittance function, it is a PRF if the following conditions are satisfied:

1. All the coefficients of $P(s)$ and $Q(s)$ are real and positive.
2. $Q(s)$ is a Hurwitz polynomial.
3. All the terms in the numerator and denominator polynomial are present. There is no missing term.
4. All the poles and zeros of $F(s)$ lie in the left half of the s-plane.

Testing of a function to be positive real or not is illustrated through a few examples.

The necessary and sufficient conditions for a rational network function $F(s)$ to be positive real (that is, physically realisable) are stated as follows:

Condition 1: $F(s)$ must not have any pole in the right-hand side of s-plane; that is, both $P(s)$ and $Q(s)$ of $F(s)=\dfrac{P(s)}{Q(s)}$ are Hurwitz.

This condition can be checked by continued fraction expansion of the odd to even parts or even to odd parts of $P(s)$ and $Q(s)$.

Condition 2: real part of F(s), that is,

Re$[F(j\omega)]\geq 0$, for all ω

or $$A(\omega^2)=M_1(j\omega)\times M_2(j\omega)-N_1(j\omega)\times N_2(j\omega)\geq 0 \text{ for all } \omega.$$

Where $M_1(s)$ and $N_1(s)$ are even and odd parts of $P(s)$ respectively. $M_2(s)$ and $N_2(s)$ are the even part and odd part of $Q(s)$ respectively.

Condition 3: This condition is tested when the poles of $F(s)$ lie on the $j\omega$-axis. This is done by using the partial fraction expansion of $F(s)$ to check whether the residues of the poles on the $j\omega$-axis are positive and real.

Example 12.7 Determine whether the function $F(s) = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14}$ is a positive real functions (PRF) and hence realisable.

Solution: Given

$$F(s) = \frac{s^2 + 6s + 5}{s^2 + 9s + 14} = \frac{P(s)}{Q(s)}$$

The following necessary conditions are first applied to the given function as

1. All the coefficients of P(s) and Q(s) are real and positive.
2. All the terms in the numerator and denominator polynomial are present. There is no missing term.

Hence, the function $F(s)$ may be a PRF.

To check whether it is truly PRF or not, we will check for the sufficient conditions.

Step 1: firstly, we will check whether $F(s)$ is Hurwitz or not

Given $$F(s) = \frac{s^2 + 6s + 5}{s^2 + 9s + 14} = \frac{P(s)}{Q(s)}$$

Let us consider $P(s)$ as follows:

$$P(s) = s^2 + 6s + 5$$

Even part of $P(s) = s^2 + 5$ and odd part $= 6s$

The continued fraction expansion is carried out for $P(s)$ and $Q(s)$ as follows:

$$6s\overline{)s^2 + 5}\left(\frac{s}{6}\right.$$

$$\underline{s^2}$$

$$5\overline{)6s}\left(\frac{6}{5}s\right.$$

$$\underline{6s}$$

x

Since all the quotient terms of $P(s)$ are positive, $P(s)$ is Hurwitz.

Now, let us consider $Q(s)$ as in the following:

$$Q(s) = s^2 + 9s + 14$$

Even part of $Q(s) = s^2 + 14$ and the odd part of $Q(s) = 9s$

The continued fraction expansion is as follows:

$$9s\overline{)s^2+14}\left(\frac{s}{9}\right.$$
$$\underline{s^2}$$
$$14\left.\right)9s\left(\frac{9}{14}s\right.$$
$$\underline{9s}$$
$$\text{x}$$

All the quotient terms of $Q(s)$ are positive. Therefore, $Q(s)$ is also Hurwitz.

Since $P(s)$ and $Q(s)$ are both Hurwitz, $F(s)=\dfrac{P(s)}{Q(s)}$ is also Hurwitz.

Step 2: to check whether $\text{Re}|F(j\omega)| \geq 0$, for all ω, that is,
$A(\omega^2) = M_1(j\omega) \times M_2(j\omega) - N_1(j\omega) \times N_2(j\omega) \geq 0$ for all ω.

$$F(s)=\frac{P(s)}{Q(s)}=\frac{s^2+6s+5}{s^2+9s+14}$$

Now, $$P(s) = s^2 + 6s + 5$$

The even part is $M_1(s) = s^2 + 5$ and its odd part is $N_1(s) = 6s$

$$Q(s) = s^2 + 9s + 14$$

The even part is $M_2(s) = s^2 + 14$ and its odd part is $N_2(s) = 9s$

Consider $$A(\omega^2) = M_1(s) \times M_2(s) - N_1(s) \times N_2(s)\Big|_{s=j\omega}$$

or $$A(\omega^2) = (s^2+5)(s^2+14) - (6s)(9s)\Big|_{s=j\omega}$$

or $$A(\omega^2) = (s^2+5)(s^2+14) - 54s^2\Big|_{s=j\omega}$$
$$= s^4 + 14s^2 + 5s^2 + 70 - 54s^2\Big|_{s=j\omega}$$
$$= s^4 + 19s^2 + 70 - 54s^2\Big|_{s=j\omega}$$
$$= s^4 - 35s^2 + 70\Big|_{s=j\omega}$$
$$= (j\omega)^4 - 35(j\omega)^2 + 70$$
$$= j^4\omega^4 - 35j^2\omega^2 + 70$$
$$= 1\cdot\omega^4 - 35(-1)\omega^2 + 70 \quad [\because j^2 = -1, j^4 = 1]$$
$$= \omega^4 + 35\omega^2 + 70$$

Since $A(\omega^2) \geq 0$ for all ω, the given function is a positive real functions (PRF) and the function is physically realisable network.

Example 12.8 Test whether $F(s)=\dfrac{s^2+\frac{3}{4}s+\frac{3}{4}}{s^2+s+4}$ is a positive real functions (PRF) and represents a physically realisable network.

Solution:

Step 1: check whether $F(s)$ is Hurwitz or not

Now,

$$F(s) = \frac{s^2 + \frac{3}{4}s + \frac{3}{4}}{s^2 + s + 4} = \frac{P(s)}{Q(s)}$$

Now, consider $P(s)$ as in the following:

$$P(s) = s^2 + \frac{3}{4}s + \frac{3}{4}$$

The even part $M_1(s) = s^2 + \frac{3}{4}$ and its odd part $N_1(s) = \frac{3}{4}s$

Continued fraction expansion is as follows:

$$\begin{array}{l} \frac{3}{4}s \overline{)s^2 + \frac{3}{4}} \left(\frac{4}{3}s \right. \\ \quad \underline{s^2} \\ \quad\quad \frac{3}{4} \Big) \frac{3}{4}s \left(s \right. \\ \quad\quad\quad \underline{\frac{3}{4}s} \\ \quad\quad\quad\quad \text{x} \end{array}$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Now, take $Q(s) = s^2 + s + 4$

The even part is $M_2(s) = s^2 + 4$ and its odd part is $N_2(s) = s$

Continued fraction expansion is given as follows:

$$\begin{array}{l} s \overline{)s^2 + 4} \left(s \right. \\ \quad \underline{s^2} \\ \quad\quad 4 \Big) s \left(\frac{s}{4} \right. \\ \quad\quad\quad \underline{s} \\ \quad\quad\quad\quad \text{x} \end{array}$$

Quotient terms are positive ad $Q(s)$ is also Hurwitz.

Since both $P(s)$ and $Q(s)$ are Hurwitz, and therefore, $F(s)$ is also Hurwitz.

Step 2: consider $A(\omega^2) = M_1(s)M_2(s) - N_1(s)N_2(s)\Big|_{s=j\omega}$

$$= \left(s^2 + \frac{3}{4}\right)(s^2 + 4) - \left(\frac{3}{4}s\right)(s)\Big|_{s=j\omega}$$

$$= s^4 + 4s^2 + \frac{3}{4}s^2 + 3 - \frac{3}{4}s^2 \Big|_{s=j\omega}$$

$$= s^4 + 4s^2 + 3\Big|_{s=j\omega}$$

$$A(\omega^2) = (j\omega)^4 + 4(j\omega)^2 + 3$$

$$= j^4\omega^4 + 4j^2\omega^2 + 3$$

or,
$$A(\omega^2) = \omega^4 - 4\omega^2 + 3 \quad \text{(i)}$$

Since all terms in $A(\omega^2)$ are not positive, we have to check whether $A(\omega^2) \geq 0$ for all values of ω.

Let us apply Sturm test to check $A(\omega^2) \geq 0$. (It is obtained by considering the derivative of $P_0(x)$.)

For this, we substitute $\omega^2 = x$, in (i) as

Then, $P_0(x) = x^2 - 4x + 3$

Derivative of $P_0(x)$, that is, $P_1(x) = 2x - 4$

Now, dividing $P_0(x)$ by $P_1(x)$, we get the following:

$$2x - 4\overline{)x^2 - 4x + 3}\left(\frac{x}{2} + 1\right.$$

$$\underline{x^2 - 2x}$$

$$-2x + 3$$

$$\underline{2x - 4}$$

$$\underline{-1} = -P_2(x)$$

$P_0(x) = x^2 - 4x + 3$

$P_1(x) = 2x - 4$ and $P_2(x) = 1$

Now, we find the sign changes of $P_0(x)$, $P_1(x)$, $P_2(x)$ as x changes from 0 to ∞

Limits	Sign of P_0(x)	Sign of P_1(x)	Sign of P_2(x)	Sign Changes
$x = 0$	positive	negative	positive	2
$x = \infty$	positive	positive	positive	No sign change

Since there are two sign changes, $A(\omega^2)$ is always positive for all values of ω. The given function is not PRF.

(overall sign changes = 2 – 0 = 2)

Example 12.9 Test whether the function $A(\omega^2) = 3\omega^4 - 12\omega^2 + 4$ is positive real.

Solution: Given $A(\omega^2) = 3\omega^4 - 12\omega^2 + 4$. To check whether $A(\omega^2) \geq 0$ for all values of ω, let us apply Sturm test.

Substitute $\omega^2 = x$.

Then, $P_0(x) = 3x^2 - 12x + 4$

and $P_1(x) = 6x - 12$ (derivative of $P_0(x)$)

Now, divide $P_0(x)$ by $P_1(x)$

$$6x-12\overline{)3x^2-12x+4}\left(\frac{x}{2}-1\right.$$

$$\underline{3x^2-6x}$$

$$-6x+4$$

$$\underline{6x+12}$$

$$\underline{\quad -8} \Rightarrow -P_2$$

Now,

$$P_0(x) = 3x^2 - 12x + 4$$

$$P_1(x) = 6x - 12$$

$$P_2(x) = +86$$

We find sign changes of $P_0(x)$, $P_1(x)$, $P_2(x)$ for $x = 0$ and $x = \infty$.

	$P_0(x)$	$P_1(x)$	$P_2(x)$	Sign Changes
$x = 0$	positive	negative	positive	2
$x = \infty$	positive	positive	positive	No sign change

Since there are two sign changes, the function is not positive real.
(Overall sign changes = 2 – 0 = 2)

Example 12.10 Test whether $Y(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ is a positive real function.

Solution:

Step 1: let us check whether $Y(s)$ is Hurwitz or not.

Now,

$$Y(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \frac{P(s)}{Q(s)}$$

Consider $P(s)$ as in the following:

$$P(s) = 2s^3 + 2s^2 + 3s + 2$$

Its odd part $N_1(s) = 2s^3 + 3s$ and even part $M_1(s) = 2s^2 + 2$

Now,

$$2s^2+2\overline{)2s^3+3s}\left(s\right.$$

$$\underline{2s^3+2s}$$

$$s\,)2s^2+2\left(2s\right.$$

$$\underline{2s^2}$$

$$2\,)s\left(\frac{s}{2}\right.$$

$$\underline{\quad s}$$

$$\text{x}$$

Since all the quotients are positive, $P(s)$ is Hurwitz.

Now, consider $Q(s) = s^2 + 1$

Its even part $M_2(s) = s^2 + 1$ and there is no odd part; that is, $N_2(s) = 0$

Therefore, odd part (derivative of $M_2(s)$) $= 2s$

Now,

$$
\begin{array}{r}
\dfrac{s}{2} \\
2s\overline{)s^2+1}\left(\right. \\
\underline{s^2} \\
1 \\
\underline{2s} \\
\text{x}
\end{array}
\qquad
\begin{array}{l}
\left. \right) 2s \left(2s \right. \\
\\
\left. \right)
\end{array}
$$

Since all the quotients are positive, $Q(s)$ is also Hurwitz.

Now, both $P(s)$ and $Q(s)$ are Hurwitz, and therefore, $Y(s)$ is also Hurwitz.

Step 2: let us find $A(\omega^2) = M_1(s)\, M_2(s) - N_1(s)N_2(s)\Big|_{s=j\omega}$

$$= (2s^2 + 2)(s^2 + 1) - (2s^3 + 3s)(0)\Big|_{s=j\omega}$$

$$= 2s^4 + 2s^2 + 2s^2 + 2 - 0 - 0\Big|_{s=j\omega}$$

$$= 2s^4 + 4s^2 + 2\Big|_{s=j\omega}$$

Now,

$$A(\omega^2) = 2(j\omega)^4 + 4(j\omega)^2 + 2$$
$$= 2j^4\omega^4 + 4j^2\omega^2 + 2$$
$$= 2\omega^4 - 4\omega^2 + 2$$

Let us apply Sturm test to check whether $A(\omega^2) \geq 0$ or not

Substitute $\omega^2 = x$

Then, $P_0(x) = 2x^2 - 4x + 2$

and $P_1(x) = 4x - 4$

Now, divide $P_0(x)$ by $P_1(x)$, we get the following:

$$
\begin{array}{r}
\dfrac{x}{2} - \dfrac{1}{2} \\
4x - 4\overline{)2x^2 - 4x + 2}\left(\right. \\
\underline{2x^2 - 2x} \\
-2x + 2 \\
\underline{-2x + 2} \\
0
\end{array}
\qquad
\begin{array}{l}
\left. \right) \\
\\
\left. \right) \\
\\
\left. \right) \\
\\
= -P_2(x)
\end{array}
$$

Therefore,

$$P_0(x) = 2x^2 - 4x + 2$$
$$P_1(x) = 4x - 4$$
$$P_2(x) = -0$$

Limits	Sign of $P_0(x)$	Sign of $P_1(x)$	Sign of $P_2(x)$	Sign Changes
$x = 0$	positive	negative	negative	1
$x = \infty$	positive	positive	negative	1

Therefore, overall sign changes = 1 − 1 = 0.

Since there is no overall sign change, the given function is a PRF, that is, positive real functions.

12.4 SYNTHESIS OF NETWORKS BY FOSTER'S AND CAUER'S METHODS

The Foster's method of network synthesis uses the partial fraction expansion of the driving point immittance function. Foster is of two types: Foster form-I and Foster form-II. When the driving point function is an impedance, it is referred to as Foster form-I. Foster form-II is used for driving point admittance function.

The Cauer's methods employ continued expansion approach to synthesise a given immittance function. In Cauer form-I, the terms of both numerator and denominator are arranged in descending degree of s. In Cauer form-II, the terms in the numerator and denominator polynomials of the driving point immittance function are arranged in ascending order.

Foster and Cauer methods of network synthesis are presented in details.

12.5 FOSTER AND CAUER FORMS

12.5.1 Synthesis of *R–C* Network

The impedance $Z(s)$ of *R–C* network is of the type as in the following:

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)}{(s+\sigma_2)(s+\sigma_4)}$$

Further, for *R–C* impedance function, $Z(0) \geq Z(\infty)$.

12.5.2 Properties of the *R–C* Impedance or *R–L* Admittance Function

1. The poles and zeros of the $Z_{RC}(s)$ or $Y_{RL}(s)$ lie on the negative real axis of the complex s-plane.
2. The poles and zeros are interlacing, that is, they alternate along the negative real axis.
3. The poles and zeros are simple. There are no multiple poles and zeros.
4. $Z_{RC}(0) > Z_{RC}(\infty)$

12.5.3 Foster Form-I of *R–C* Network

It is obtained by using the partial fraction expansion of $Z_{RC}(s)$.

The partial fraction expansion of $Z_{RC}(s)$ is given as follows:

$$Z_{RC}(s) = \frac{P_0}{s} + \frac{P_2}{s+\sigma_2} + \frac{P_4}{s+\sigma_4} + \cdots + P_\infty$$

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

represents capacitor | represents parallel comb. of R and C | parallel comb. of R and C | represents resistor

$$C_0 = \frac{1}{P_0} \qquad R_2 = \frac{P_2}{\sigma_2} \qquad R_4 = \frac{P_2}{\sigma_4} \qquad R_\infty = P_\infty$$

$$C_2 = \frac{1}{P_2} \qquad C_4 = \frac{1}{P_4}$$

Accordingly, the Foster form-I of R–C network is shown in Figure 12.1.

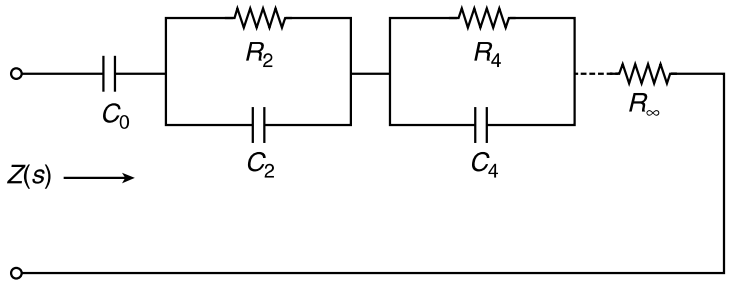

Figure 12.1 *Foster Form-I of R–C Network*

12.5.4 Foster Form-II of *R–C* Network

It is obtained from the following:

$$Y_{RC}(s) = \frac{1}{Z_{RC}(s)}$$

$$Y_{RC}(s) = P_0 + \frac{P_2 s}{s+\sigma_2} + \cdots + P_\infty s$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

represents resistor | represents series comb. of R and C | represents capacitance $\Downarrow$ $\Downarrow$

$$R_o = \frac{1}{P_0} \qquad R_2 = \frac{1}{P_2} \qquad C_\infty = P_\infty$$

$$C_2 = \frac{P_2}{\sigma_2}$$

Foster from-II of *R–C* network is shown in Figure 12.2.

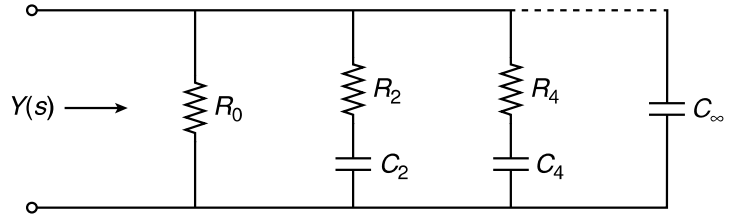

Figure 12.2 *Foster Form-II of R–C network*

12.5.5 Cauer Forms of *R–C* Network

The *R–C* network for Cauer form-I and Cauer form-II is shown in Figure 12.3 (a) and (b), respectively. Note that in Cauer form-I resistors are in series and capacitors are in parallel. In Cauer form-II capacitors are in series while the resistors are in parallel. Method of calculation of their values will be illustrated through an example.

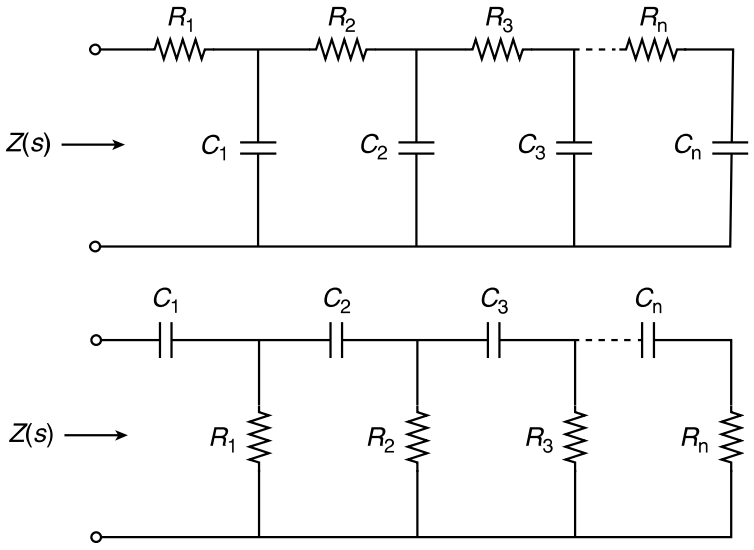

Figure 12.3 *(a) Cauer Form-I of R–C Network and (b) Cauer Form-II of R–C Network*

12.5.6 Synthesis of *R–L* Network

General driving point impedance of *R–L* network is given by the following:

$$Z_{\mathrm{RL}} = \frac{H(s+\sigma_1)(s+\sigma_3)(s+\sigma_5)....}{(s+\sigma_2)(s+\sigma_4)(s+\sigma_6).....}$$

In the case of *R–L* network, $Z_{\mathrm{RL}}(\infty) > Z_{\mathrm{RL}}(0)$.

12.5.7 Properties of *R–L* Impedance Function/*R–C* Admittance Function

1. The poles and zeros are simple. There are no multiple poles or zeros.
2. The poles and zero interlace (that is, alternate) each other along the negative real axis.

3. The poles and zeros lie on the negative real axis of the complex s-plane.
4. $Z(\infty) > Z(0)$

12.5.8 Foster Form-I of *R–L* Network

The partial fraction expansion of $Z_{RL}(s)$ is given as follows:

$$Z(s) = P_o + \frac{P_2 s}{s+\sigma_2} + \frac{P_4 s}{s+\sigma_4} + \cdots + P_\infty s$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$R_0 = P_0 \quad R_2 = P_2 \quad R_4 = P_4 \quad L_\infty = P_\infty$$

$$L_2 = \frac{P_2}{\sigma_2} \quad L_4 = \frac{P_4}{\sigma_4}$$

Figure 12.4 shows the Foster form-I of *R–L* network.

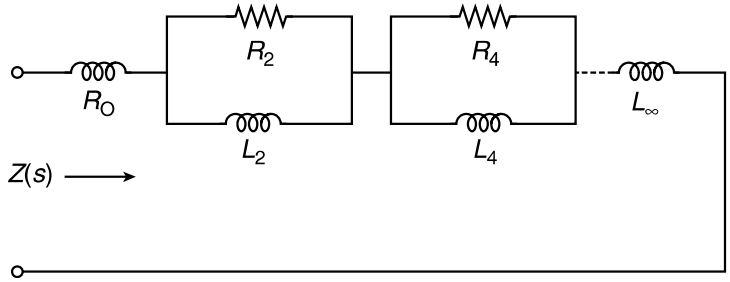

Figure 12.4 *Foster Form-I of R–L Network*

12.5.9 Foster Form-II of *R–L* Network

Consider the foster form-II of *R–L* network shown in Figure 12.5.

The network is designed by using the partial fraction method. Here, we consider the partial fraction expansion of $Y(s)$.

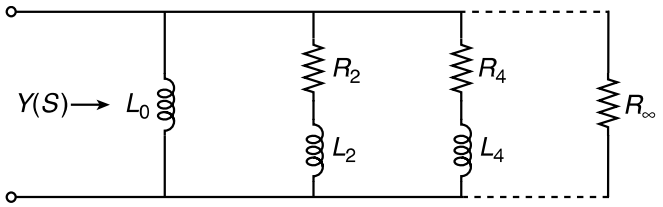

Figure 12.5 *Foster Form-II of R–L Network*

$$Y(s) = \frac{P_0}{s} + \frac{P_2}{s+\sigma_2} + \frac{P_4}{s+\sigma_4} + \cdots + P_\infty$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$L_0 = \frac{1}{P_o} \quad R_2 = \frac{\sigma_2}{P_2} \quad R_4 = \frac{\sigma_4}{P_4} \quad R_\infty = P_\infty$$

$$L_2 = \frac{1}{P_2} \quad L_4 = \frac{1}{P_4}$$

12.5.10 Cauer Form-I of *R–L* Network

Figure 12.6(a) shows the representation of cauer form-I for *R–L* network.

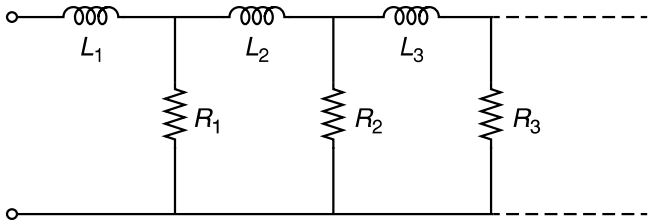

Figure 12.6(a) *Schematic Diagram Shows Cauer Form-I of R–L Network*

12.5.11 Cauer Form-II *R–L* Network

Figure 12.6(b) shows the representation of cauer form-II for *R–L* network.

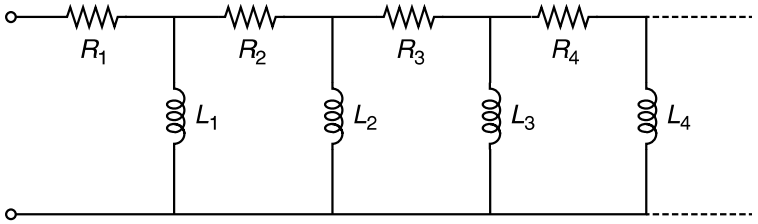

Figure 12.6(b) *Schematic Diagram Shows Cauer Form-II of R–L Network*

12.5.12 Synthesis of *L–C* Networks

LC immittance is given by

$$Z(s) = \frac{H(s^2 + \omega_1^2)(s^2 + \omega_3^2)\ldots}{s(s^2 + \omega_2^2)(s^2 + \omega_4^2)\ldots}$$

12.5.13 Properties of *L–C* Immittance

1. The immittance function (that is, $Z(s)$ or $Y(s)$) is a ratio of odd to even polynomial or even to odd polynomial.
2. The highest order terms in the numerator and denominator differ by one.
3. There is a pole or zero at origin
4. There is a pole or zero at infinity
5. The poles and zeros occur in complex conjugate pairs.

12.5.14 Foster Form-I of *L–C* Network

The partial fractions of $Z_{LC}(s)$ are given by

$$Z(s) = \frac{P_0}{s} + \frac{2P_2 s}{s^2 + \omega_2^2} + \frac{2P_4 s}{s^2 + \omega_4{}^2} + \cdots + P_\infty s$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$C_0 = \frac{1}{P_0} \quad L_2 = \frac{2P_2}{\omega_2{}^2} \quad L_4 = \frac{2P_4}{\omega_4^2} \quad L_\infty = P_\infty$$

$$C_2 = \frac{1}{2P_2} \qquad C_4 = \frac{1}{2P_4}$$

The Foster form-I of the *L–C* network is shown in Figure 12.7.

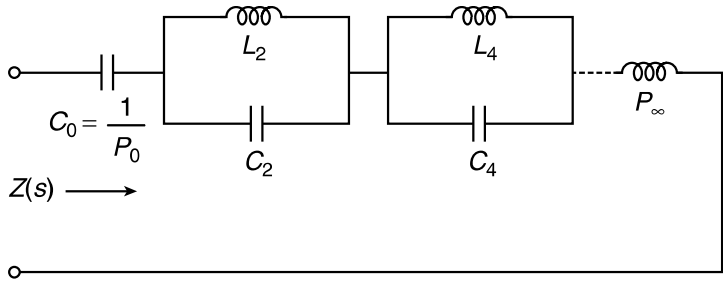

Figure 12.7 *Foster Form-I of L–C Network*

12.5.15 Foster Form-II of *L–C* Network

$$Y(s) = \frac{P_0}{s} + \frac{2P_2 s}{s^2 + \omega_2^2} + \frac{2P_4 s}{s^2 + \omega_4^2} + \ldots\ldots + P_\infty s$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$L_0 = \frac{1}{P_0} \qquad L_2 = \frac{1}{2P_2} \quad L_4 = \frac{1}{2P_4} \qquad C_\infty = P_\infty$$

$$C_2 = \frac{2P_2}{\omega_2^2} \quad C_4 = \frac{2P_4}{\omega_4^2}$$

The Foster form-II for the *L–C* network is shown in Figure 12.8.

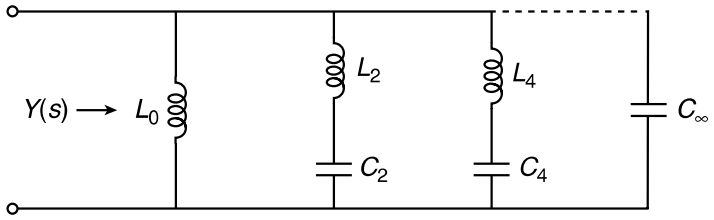

Figure 12.8 *Foster Form-II of L–C Network*

12.5.16 Cauer Form-I of *L–C* Network

The Cauer form-I of *L–C* network is shown in Figure 12.9(a) below.

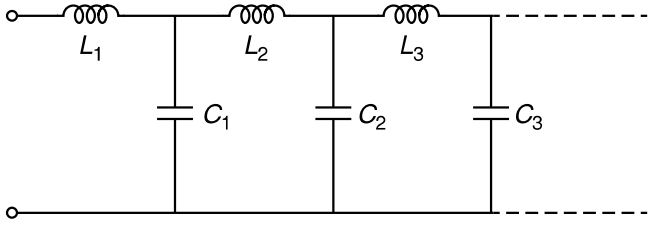

Figure 12.9(a) *Cauer Form-I of L–C Network*

12.5.17 Cauer Form-II of *L–C* Network

Cauer form-II of the *L–C* network is shown in Figure 12.9(b).

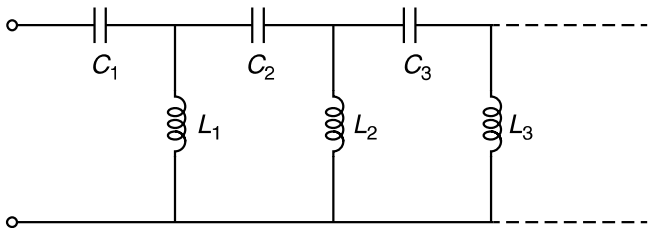

Figure 12.9(b) *Cauer Form-II of L–C Network*

Example 12.11 Show whether the following driving point functions represent a *L–C* or a *R–C* network. Give reasons for your answer.

1. $Z(s) = \dfrac{s^2 + 8s + 12}{s^2 + 6s + 5}$
2. $Z(s) = \dfrac{s^3 + 9s}{s^4 + 5s^2 + 4}$

Solution:

For function 1:

$$Z(s) = \frac{s^2 + 8s + 12}{s^2 + 6s + 5} = \frac{s^2 + 6s + 2s + 12}{s^2 + 5s + s + 5}$$

$$= \frac{(s+6)(s+2)}{(s+1)(s+5)}$$

In the above, all the poles and zeros are real, and therefore, it may be *R–C* or *R–L* impedance function. We have to check whether it is *R–C* or *R–L*. For this, we have to find $Z(\infty)$.

$$Z(s) = \frac{(s+6)(s+2)}{(s+1)(s+5)} = \frac{s\left(1+\frac{6}{s}\right)s\left(1+\frac{2}{5}\right)}{s\left(1+\frac{1}{s}\right)s\left(1+\frac{5}{s}\right)}$$

$$= \frac{\left(1+\frac{6}{s}\right)\left(1+\frac{2}{s}\right)}{\left(1+\frac{1}{s}\right)\left(1+\frac{5}{s}\right)}$$

Substituting $s = \infty$, we get the following:

$$Z(\infty) = \frac{(1+0)(1+0)}{(1+0)(1+0)} = 1$$

That is, $Z(\infty) = 1$

Now,
$$Z(0) = \frac{(0+6)(0+2)}{(0+1)(0+5)} = \frac{12}{5}$$

Thus, $Z(0) > Z(\infty)$. Therefore, the given impedance function is of *R–C* type. The property of *R–C* impedance as stated in section 12.5.2, is that $Z_{RC}(0) > Z_{RC}(\infty)$.

For function 2:
$$Z(s) = \frac{s^3 + 9s}{s^4 + 5s^2 + 4}$$
$$= \frac{s(s^2+9)}{s^4 + 4s^2 + s^2 + 4}$$
$$= \frac{s(s^2+9)}{s^2(s^2+4) + 1(s^2+4)}$$
$$= \frac{s(s^2+9)}{(s^2+1)(s^2+4)}$$
$$\Downarrow$$

Since $Z(s)$ has quadratic factors and there is a difference of one between degree of numerator and denominator, it can be written as follows:

$Z(s) = \dfrac{s^3 + 9s}{s^4 + 5s^2 + 4}$ is of *L–C* type. (See properties of *L–C* impedance section in 12.5.13)

Example 12.12 Synthesise all the four forms of the *R–C* driving point function represented as
$$Z_{RC}(s) = \frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution: To synthesis the Foster form-I circuit, let us consider the given driving point function as
$$Z(s) = \frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

The partial fraction expansion of $Z(s)$ will be as follows:
$$= 2 + \frac{A}{s+1} + \frac{B}{s+3} \tag{12.1}$$

[$\because$ degree of numerator and denominator is same]

Let us find the values of A and B
$$\frac{2(s+2)(s+4)}{(s+1)(s+3)} = 2 + \frac{A}{s+1} + \frac{B}{s+3}$$
$$\frac{2(s+2)(s+4)}{(s+1)(s+3)} = \frac{2(s+1)(s+3) + A(s+3) + B(s+1)}{(s+1)(s+3)}$$

or $$2(s+2)(s+4) = 2(s+1)(s+3) + A(s+3) + B(s+1) \quad (12.2)$$

To find A, substitute $s = -1$ in equation (12.2), we get

$$2(-1+2)(-1+4) = 0 + A(-1+3) + 0$$

or $$2(1)(3) = A(2)$$

or $$A = 3$$

To find the value of B, substitute $s = -3$ in equation (12.2)

$$2(-3+2)(-3+4) = 0 + 0 + B(-3+1)$$

or $$2(-1)(1) = B(-2)$$

or $$-2 = -2B$$

or $$B = 1$$

Now, substituting the values of A and B in equation (12.1), we get the following form:

$$Z(s) = 2 + \frac{3}{s+1} + \frac{1}{s+3}$$

$$Z(s) = \frac{P_2}{s+\sigma_2} + \frac{P_4}{s+\sigma_4} + P_\infty$$

In this case, term $\frac{P_0}{s}$ is missing, and therefore, C_0 is absent. (For reference, see Figure 12.1)

Term $\frac{3}{s+1}$ represents the parallel combination of R_2 and C_2.

where $$R_2 = \frac{P_2}{\sigma_2} = \frac{3}{1} = 3\,\Omega$$

$$C_2 = \frac{1}{P_2} = \frac{1}{3}\,\text{F}$$

General form of Foster form-I is represented as follows. The corresponding network is shown in Figure 12.10.

$$Z(s) = \frac{P_0}{s} + \frac{P_2}{s+\sigma_2} + P_\infty$$

$\downarrow$ $\downarrow$ $\downarrow$

represents capacitor $C_0 = \frac{1}{P_0}$

represents R and C in parallel $R_2 = \frac{P_2}{\sigma_2}$, $C_2 = \frac{1}{P_2}$

represents resistor $R_\infty = P_\infty$

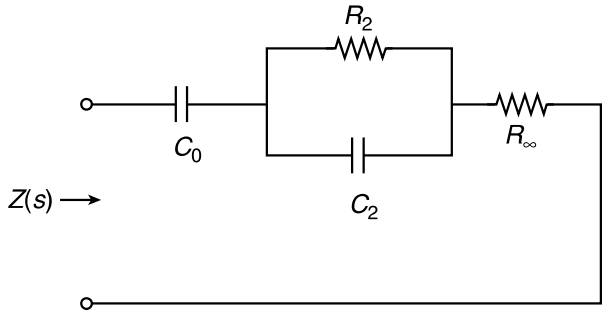

Figure 12.10

Term $\frac{1}{s+3}$ represents the parallel combination of R_4 and C_4.

where
$$R_4 = \frac{P_4}{\sigma_4} = \frac{1}{3}\Omega$$
$$C_4 = \frac{1}{P_4} = \frac{1}{1} = 1\text{F}$$

Required Foster form-I is shown in Figure 12.11. Note that in the network, C_0 is absent.

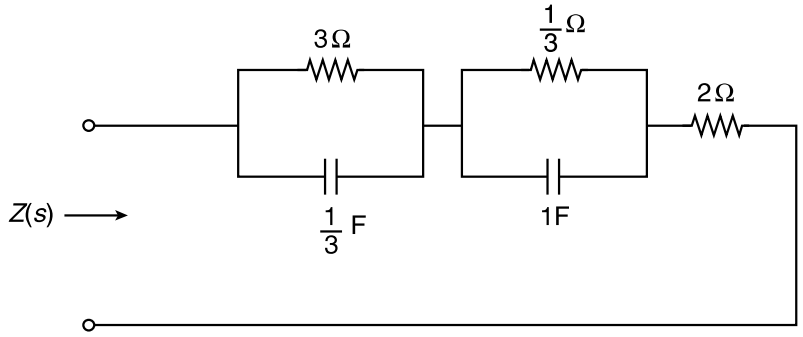

Figure 12.11

Foster form-II circuit is developed from $Y(s)$ of the given impedance function as

$$Y(s) = \frac{1}{Z(s)} = \frac{(s+1)(s+3)}{2(s+2)(s+4)}$$
$$= \frac{1}{2} \cdot \frac{(s+1)(s+3)}{(s+2)(s+4)} \tag{12.3}$$

Since the degree of numerator and denominator is same, the partial fraction will be as follows:

$$Y(s) = \frac{1}{2} + \frac{A}{s+2} + \frac{B}{s+4} \tag{12.4}$$

Let us find the values of A and B.

Now,
$$\frac{1}{2} \cdot \frac{(s+1)(s+3)}{(s+2)(s+4)} = \frac{(s+2)(s+4) + A \cdot 2(s+4) + B \cdot 2(s+2)}{2(s+2)(s+4)}$$

or
$$(s+1)\,(s+3) = (s+2)\,(s+4) + A \cdot 2(s+4) + B \cdot 2(s+2) \tag{12.5}$$

To find A, substitute $s = -2$ in equation (12.5) [$\because$ in the denominator of A factor is $s + 2$]

$$(-2+1)\,(-2+3) = 0 + A \cdot 2(-2+4) + 0$$

or
$$(-1)\,(+1) = A \cdot 2(2)$$

or
$$(-1) = 4A$$

or
$$A = \frac{-1}{4}(\text{negative})$$

Here, none of the coefficient can be negative. In such cases, we will use the partial fractions as in the following:

$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{2s(s+2)(s+4)}$$

Now, degree of denominator > degree of numerator, and therefore, partial fraction will be used as follows:

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4} \tag{12.6}$$

Value of $A = s \cdot \left.\frac{(s+1)(s+3)}{2s(s+2)(s+4)}\right|_{s=0} = \left.\frac{(s+1)(s+3)}{2(s+2)(s+4)}\right|_{s=0} = \frac{(0+1)(0+3)}{2(0+2)(0+4)}$

$$= \frac{1\times 3}{2\times 2\times 4}$$

$$= \frac{3}{16}$$

$$B = Y(s)(s+2)\big|_{s=-2}$$

$$= \left.\frac{(s+1)(s+3)}{2s(s+4)}\right|_{s=-2}$$

$$= \frac{(-2+1)(-2+3)}{2\times(-2)(-2+4)}$$

$$= \frac{1}{8}$$

$$C = (s+4)\cdot\left.\frac{(s+1)(s+3)}{2s(s+2)(s+4)}\right|_{s=-4} = \left.\frac{(s+1)(s+3)}{2s(s+2)}\right|_{s=-4}$$

$$= \frac{(-4+1)(-4+3)}{2(-4)(-4+2)}$$

$$= \frac{(-3)(-1)}{2(-4)(-2)} = \frac{3}{16}$$

$$C = \frac{3}{16}$$

Now, substituting the values of A, B and C in equation (12.6), we get the following:

$$\frac{Y(s)}{s} = \frac{\frac{3}{16}}{s} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{16}}{s+4}$$

or,

$$Y(s) = \frac{3}{16} + \frac{\frac{1}{8}s}{s+2} + \frac{\frac{3}{16}s}{s+4} \tag{12.7}$$

Here,

$$P_0 = \frac{3}{16}$$

$$P_2 = \frac{1}{8}, \sigma_2 = 2$$

$$P_4 = \frac{3}{16}, \sigma_4 = 4$$

General partial fractions of $Y_{RC}(s)$ are

$$Y_{RC}(s) = P_0 + \frac{P_2 s}{s+\sigma_2} + \frac{P_4 s}{s+\sigma_4} + \cdots + P_\infty \tag{12.8}$$

Therefore, comparing equations (12.7) and (12.8), we get the following:

term $\frac{3}{16}$ will represent resistor of value $R_0 = \frac{1}{P_0} = \frac{16}{3}\Omega$

Term $\frac{\frac{1}{8}s}{s+\sigma_2}$ represents series combination of R_2 and C_2

where $R_2 = \frac{1}{P_2} = 8\,\Omega$

$$C_2 = \frac{P_2}{\sigma_2} = \frac{1}{16}\text{F}$$

Term $\frac{\frac{3}{16}s}{s+4}$ represents series combination of R_4 and C_4.

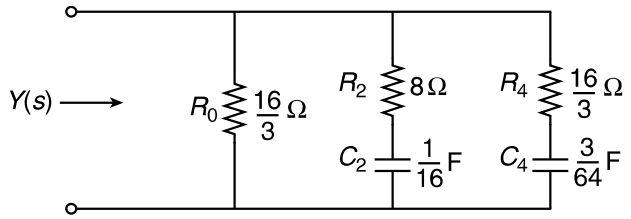

Figure 12.12

where $R_4 = \frac{1}{P_4} = \frac{16}{3}\Omega$

$$C_4 = \frac{P_4}{\sigma_4} = \frac{3}{64}\text{F}$$

Therefore, required Foster form-II, is shown in Figure 12.12.

In Cauer form-I, we arrange numerator and denominator of $Z(s)$ in descending power of s and then carry out continued fraction expansion to identify the circuit elements from the quotient terms.

Given

$$Z(s) = \frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

$$= \frac{2(s^2+4s+2s+8)}{s^2+s+3s+3} = \frac{2s^2+12s+16}{s^2+4s+3}$$

$$s^2+4s+3\overline{)2s^2+12s+16}\left(2 \Rightarrow R_1(\text{Constant term represents resistor}) \rightarrow Z_1\right.$$

$$\underline{2s^2+8s+6}$$

$$4s+10\;)\,s^2+4s+3\left(\frac{s}{4} \Rightarrow C_1 \rightarrow Y_1\right.$$

$$\underline{s^2+\frac{5}{2}s}$$

$$\frac{3}{2}s+3\;)\,4s+10\left(\frac{8}{3} \Rightarrow R_2 \rightarrow Z_2\right.$$

$$\begin{array}{l} \underline{4s+8} \\ 2\overline{)\dfrac{3}{2}s+3}\left(\dfrac{3}{2}s \Rightarrow C_2 \rightarrow Y_2\right. \\ \quad \underline{\dfrac{3}{2}s} \\ \quad\quad 3\overline{)2}\left(\dfrac{2}{3} \Rightarrow R_3 \rightarrow Z_3\right. \\ \quad\quad\quad \underline{2} \\ \quad\quad\quad x \end{array}$$

Note that in Cauer form-I realization of the circuit element from $Z(s)$, quotient terms with s in the numerator represents the capacitor. The first quotient element will be Z_1 since we started with $Z(s)$.

We know, in Cauer form-I of R–C, resistors are in series and capacitors are in shunt, that is, parallel.

Therefore, required Cauer form-I is shown in Figure 12.13.

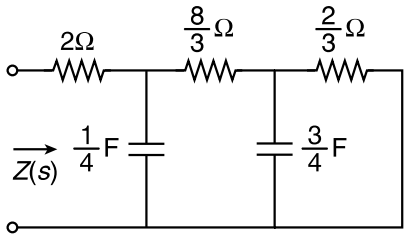

Figure 12.13

For Cauer from-II , we arrange the numerator and denominator in the ascending power of 's' so that

$$Z(s) = \frac{16+12s+2s^2}{3+4s+s^2}$$

Now,

$$\begin{array}{l} 3+4s+s^2\overline{)16+12s+2s^2}\left(\dfrac{16}{3}\right. \\ \quad \underline{16+\dfrac{64}{3}s+\dfrac{16}{3}s^2} \\ \quad\quad -\dfrac{28}{3}s+\dfrac{10}{3}s^2\Big)3+4s+s^2\left(-\dfrac{9}{285}\right. \\ \quad\quad\quad\quad\quad \downarrow \\ \quad\quad\quad\quad\quad \underline{3+\dfrac{30}{28}s} \quad \text{here the quotient is negative} \end{array}$$

Since we got a negative quotient as above, we will have to find Cauer form-II circuit by using the function $Y(s)$

$$Y(s) = \frac{1}{Z(s)} = \frac{3+4s+s^2}{16+12s+2s^2}$$

Let us find continued expansion

$$16+12s+2s^2\Big)3+4s+s^2\Big(\frac{3}{16}\Rightarrow\frac{1}{R_1}=\frac{1}{Y_1}$$

$$\underline{3+\frac{9}{4}s+\frac{3}{8}s^2}$$

$$\frac{7}{8}s+\frac{5}{8}s^2\Big)16+12s+2s^2\Big(\frac{64}{7s}\Rightarrow\frac{1}{C_1}=\frac{1}{Z_1}$$

$$\underline{16+\frac{40}{7}s}$$

$$\frac{44}{7}s+2s^2\Big)\frac{7}{4}s+\frac{5}{8}s^2\Big(\frac{49}{176}\Rightarrow\frac{1}{R_2}\Rightarrow\frac{1}{Y_2}$$

$$\underline{\frac{7}{4}s+\frac{49}{88}s^2}$$

$$\frac{6}{88}s^2\Big)\frac{44}{7}s+2s^2\Big(\frac{44\times88}{42s}\Rightarrow\frac{1}{C_2}=\frac{1}{Z_2}$$

$$\underline{\frac{44}{7}s}$$

$$2s^2\Big)\frac{6}{88}s^2\Big(\frac{3}{88}\Rightarrow\frac{1}{R_3}=\frac{1}{Y_3}$$

$$\underline{\frac{6}{88}s^2}$$

x

First element will be $\frac{1}{Y_1}$. In Cauer form-II, capacitors are in series and resistors are in parallel, as shown in Figure 12.14.

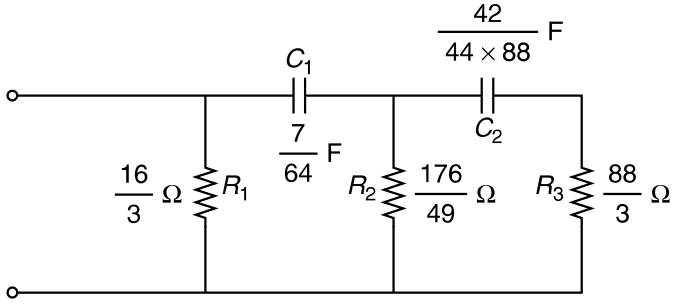

Figure 12.14 *Cauer Form-II*